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New Class of Solutions of the Korteweg-de Vries-Burgers Equation

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Abstract—We describe a new class of solutions of the KdVB equation which demonstrates chaotic behaviour.

Keywords—Dynamical chaos, Phase transition, Soliton theory.

1. INTRODUCTION

We discuss below the Korteweg-de Vries-Burgers (KdVB) equation, arising in problems where nonlinearity is present together with dispersion and dissipation, caused, e.g., by viscosity. With the dissipation neglected, the KdVB equation yields the famous Korteweg-de Vries (KdV) equation. For a long period of time, the KdV equation was considered to have only regular (non-chaotic) solutions due to its complete integrability [1]. In 1992, it was shown that this is not valid in the vicinity of singularities of the coefficients of the KdV equation. This situation takes place in many realistic cases [2]. Below, we describe the unknown class of KdVB solutions, which demonstrates chaotic behaviour. We treat these solutions both analytically and numerically in relation to the problem of phase transitions.

Our treatment is based on the assumption that the phenomena of first-kind phase transitions and transitions to dynamical chaos in dynamical systems have many common features and are therefore connected more closely than it seemed before. We consider a ferroelectric system for which the dynamical methods are well developed [3].

The methods and results of our study are of general interest because the Korteweg-de Vries-Burgers equation we use to describe our model appears in a great number of other applications.

We consider the KdVB equation that arises in the theory of ferroelectricity [4]:

$$
\begin{gathered}
P_\tau^{(1)} + AP^{(1)}P_\xi^{(1)} + DP_{\xi\xi}^{(1)} + BP_{\xi\xi\xi}^{(1)} = 0, \\
A = \frac{2\alpha}{K'(u)}, \qquad D = \frac{u}{K'(u)}, \qquad B = -\frac{u^2}{K'(u)},
\end{gathered} \tag{1}
$$

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$$K'(u) = \frac{dK(u)}{du},$$
$$K(u) = \frac{\omega_p^2 u^2}{c^2 - u^2} - \omega_0^2 - 2\alpha P_0 = 0. \qquad (1)(\text{cont.})$$

Here $P^{(1)}$ is the first term of the series expansion of the polarization P with respect to small attenuation coefficient σ; $K(u) = 0$ is the dispersive equation for wave velocity u in long-wave limit; $P_0 \simeq -\omega_0^2/\alpha$ is the equilibrium value of P; ω_p, ω_0 are the characteristic frequencies of the problem, corresponding to the optical and the acoustic branches of the spectrum; $\xi = \sigma(z - ut)$ and $\tau = \sigma^3 t$ are the scaled coordinate and time, z and t are the real ones; c is the velocity of light in the system; α is a coefficient determined by the nonlinear properties of the system.

As it was previously shown in [2], for 2π-periodic traveling wave solutions $p = P^{(1)}(\xi - v\tau)$, the attenuation-free equation (1) leads to Hammerstein's nonlinear integral equation

$$p(\theta) = -\int_0^{2\pi} G(\theta, \lambda)\left[(Av - 1)p(\lambda) - \frac{1}{2}Ap^2(\lambda)\right] d\lambda, \qquad \theta = \frac{\xi - v\tau}{\sqrt{AB}}, \qquad (2)$$

where

$$G(\theta, \lambda) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos n(\theta - \lambda)}{n^2 + 1}$$

is the 2π-periodic Green function for the equation $p_{\theta\theta} - p = -\delta(\theta - \lambda)$. Nontrivial solutions of equation (2) arise at bifurcation points, defined by the equation $Av = -n^2$, where the n^{th} harmonic of the solution grows abruptly [5]. The sequence of these bifurcation points is infinite and has an accumulation point ($n \to \infty$) at $T = T_c$, where T_c is the Curie temperature. These bifurcation points can be found from the set of equations $K(u) = 0$ and $K'(u) = 0$. This proves the hypothesis that the disorder appearing at $T > T_c$ has dynamical nature, as in many other cases, for example, revealing the Feigenbaum's cascade [6].

2. ANALYSIS OF EQUATIONS

In this section, we consider the behaviour of the system beyond the point of phase transition ($T > T_c$) and demonstrate it to have features inherent to dynamical chaos. Close to the Curie point, $u \ll c$ and $K(u) \simeq \omega_0^2 + \omega_p^2 u^2/c^2$. The equation $K(u) = 0$ has pure imaginary solutions $u = \pm ic\omega_0/\omega_p = \pm iu_i$ when $T > T_c$ and

$$\omega_0^2 = \frac{{\omega_p}^2}{2\pi}\frac{T - T_c}{\chi},$$

where χ is the Curie-Weiss constant [3]. The quantity $K'(u) = \pm 2i\omega_p\omega_0/c$ becomes imaginary too. Therefore, the coordinate $\xi = \sigma(z - ut)$ contains an imaginary contribution when $T > T_c$. To eliminate this imaginary part, we introduce the new coordinates $\tau' = \tau$ and $\xi' = \xi + i\sigma u_i t = \xi + iu_i\tau/\sigma^2$. Now equation (1) takes the form

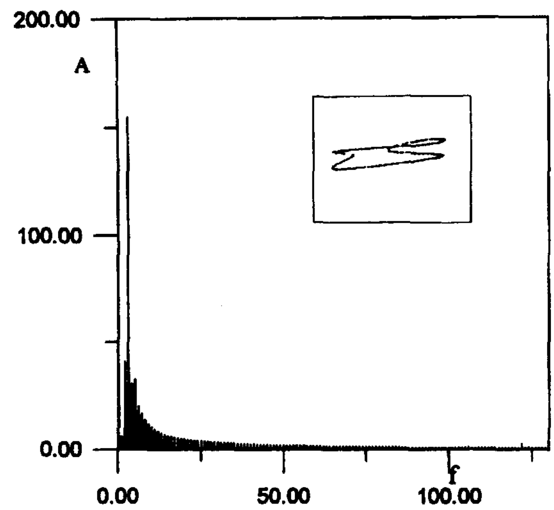
$$P_\tau^{(1)} + i\frac{c\omega_0}{\sigma^2\omega_p}P_{\xi'}^{(1)} - i\frac{\alpha c}{\omega_p\omega_0}P^{(1)}P_{\xi'}^{(1)} + \frac{c^2}{2\omega_p^2}P_{\xi'\xi'}^{(1)} - \frac{ic^3\omega_0}{2\omega_p^3}P_{\xi'\xi'\xi'}^{(1)} = 0. \qquad (3)$$

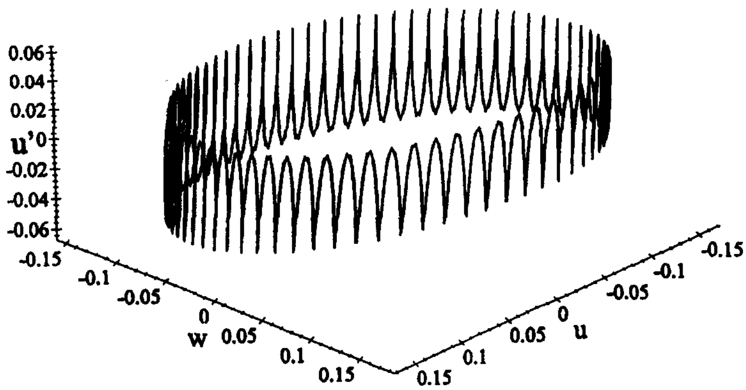
Let us examine the solutions of equation (3) of the form $P^{(1)}(\eta)$, where $\eta = (\xi' - V\tau)/c$. For the quantity

$$P = -\frac{\sigma^2\omega_p}{c\omega_0}\left[V + i\frac{\alpha c}{\omega_0\omega_p}P^{(1)}\right],$$

we obtain the following equation:

$$P_\eta - iPP_\eta - RP_{\eta\eta\eta} - i\mu P_{\eta\eta} = 0, \qquad (4)$$





(a) $\mu = 6$.

Figure 1. Three-dimensional phase portraits of a trajectory in the space of variables u, u', w; spectrum of u for numerical realization, obtained using FFT, and the projection of Poincare section of the phase manifold by the plane $u' + 0.5w' = 0$, observed in the (u, u')-plane.

A is the modulus of the spectral amplitude and f is the frequency. The scale of the frequency axis is $\Delta f = 1/\Delta\eta$, and $\Delta\eta = 20.48\pi$ is the range of η-variation.

Initial values: $u(0) = w(0) = 0.12$; $u'(0) = w'(0) = 1.2 * 10^{-5}$; $R = 0.5$ and (a) $\mu = 6$; (b) $\mu = 1.9$; (c) $\mu = 1.5$; (d) $\mu = 0.9$.

where

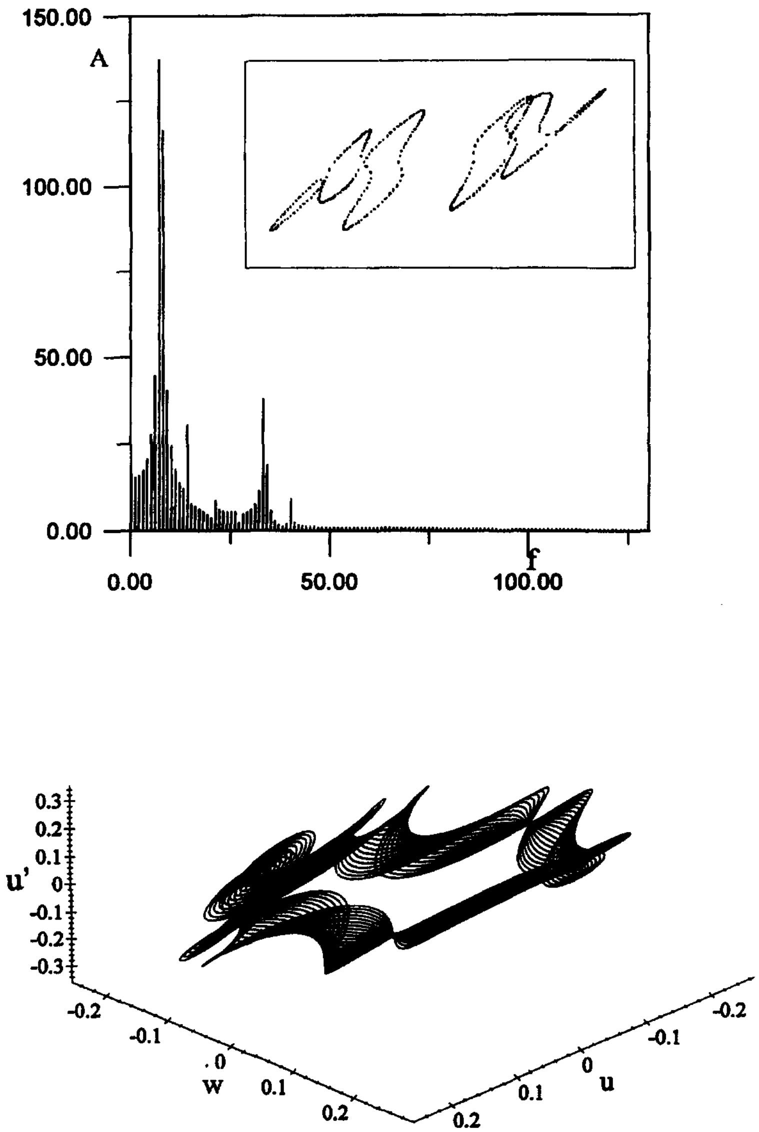
$$R = \frac{\sigma^2}{2\omega_p^2}, \qquad \mu = \frac{\sigma^2}{2\omega_0\omega_p}.$$

Equation (4) may once be integrated as follows:

$$RP_{\eta\eta} + i\mu P_\eta + i\frac{P^2}{2} - P = C, \tag{5}$$

where $C = C_1 + iC_2$ is a constant. By substituting $P = u + iw$, we can transform equation (5) to a set of two second-order equations for two functions, $u(\eta)$ and $w(\eta)$,

$$\begin{aligned} Ru_{\eta\eta} - u - uw - \mu w_\eta &= C_1, \\ Rw_{\eta\eta} - w - \frac{w^2}{2} + \frac{u^2}{2} + \mu u_\eta &= C_2. \end{aligned} \tag{6}$$



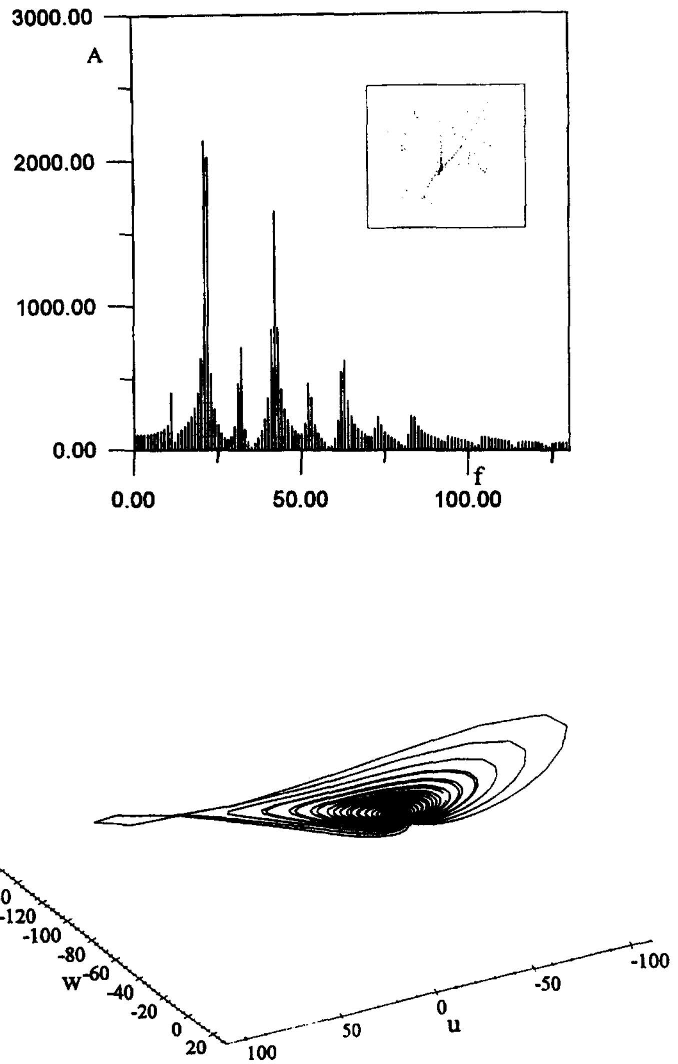
(b) $\mu = 1.9$.

Figure 1. (cont.)

Let us note that the transition from $T < T_c$ to $T > T_c$ changes the character of the equations: for $T < T_c$, the phase volume is not conserved due to dissipation, but for $T > T_c$, it is conserved as in Hamilton systems. It may be useful when one wants to determine the Kolmogorov-Sinay entropy h for the problem which is connected with the Lyapunov exponents, or the eigenvalues of linearization matrix of the equation (4). The total number of the eigenvalues is three. One of them, λ_0, is equal to zero which is due to the integrability of the equation (4). The other two eigenvalues, λ_1 and λ_2, satisfy the condition $\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 = 0$, which directly follows from the phase volume conservation. In order to calculate the Lyapunov exponents, which are just $\operatorname{Re}\lambda_{1,2}$, one should find the discrete set of values for the function [6]:

$$\Lambda\left(\vec{\rho}(0)\right) = \lim_{\substack{\eta\to\infty \\ \vec{\rho}(0)\to 0}} \eta^{-1} \ln \frac{|\vec{\rho}(\eta)|}{|\vec{\rho}(0)|}, \qquad \vec{\rho} = \vec{x} - \vec{x}_1, \tag{7}$$

where $\vec{x}(\eta)$ and $\vec{x}_1(\eta)$ are two neighbouring trajectories of equation (4). This is a complicated numerical problem, because in the general case, it has no analytical solution. One can avoid these



(c) $\mu = 1.5$.

Figure 1. (cont.)

problems, using the stationary points analysis for equation (4). There are two stationary points $P_1 = 0$ and $P_2 = -2i$ (for $C_1 = C_2 = 0$). For the linearized equation (5) near these points, one can find the eigenvalues $\tilde{\lambda}$ of the stationary linearization matrix:

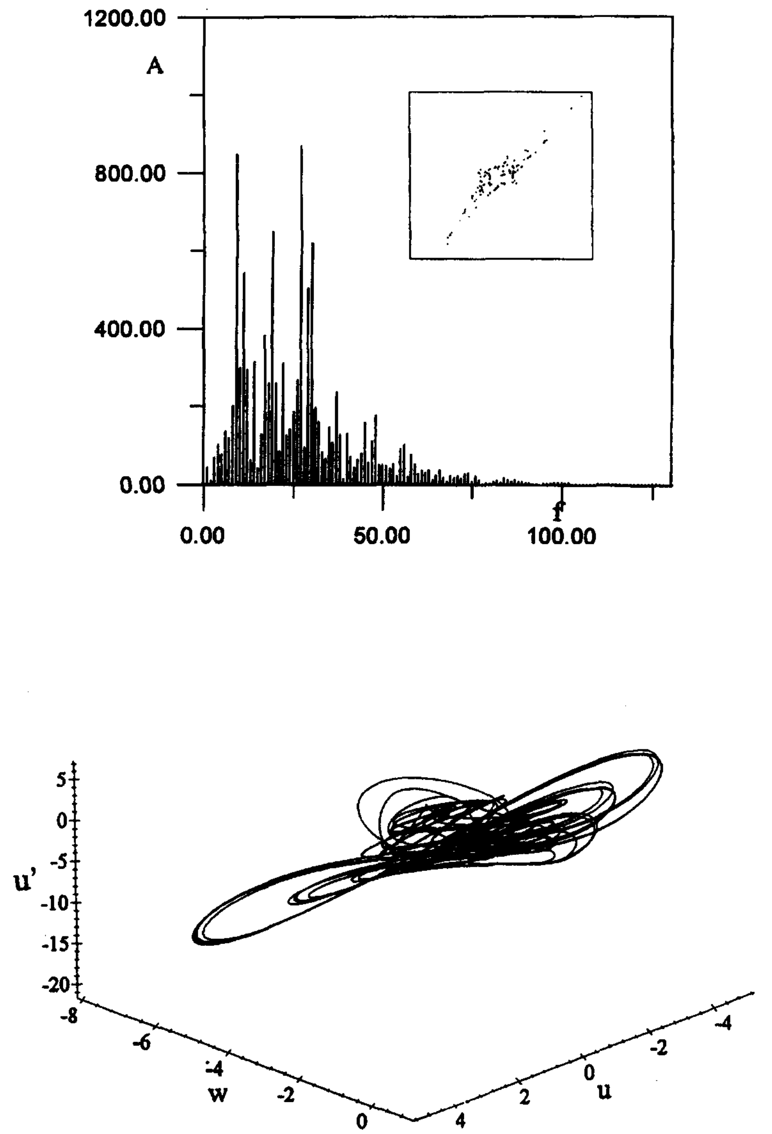
$$\tilde{\lambda}_{1,2}^{(1)} = -i\frac{\mu}{2R} \pm \sqrt{\frac{1}{R} - \left(\frac{\mu}{2R}\right)^2} \qquad \text{(saddle-focus or center)},$$

$$\tilde{\lambda}_{1,2}^{(2)} = i\left[\frac{-\mu}{2R} \pm \sqrt{\frac{1}{R} + \left(\frac{\mu}{2R}\right)^2}\right] \qquad \text{(center)}.$$

Let us use the definition (7) only for trajectories converging to the stationary point P_1. For these trajectories, the linearization along the trajectory does not differ from the one near the stationary point, and $\operatorname{Re}\lambda_1 = \operatorname{Re}\tilde{\lambda}_1$. For the opposite class of trajectories (diverging), one may get $\operatorname{Re}\lambda_2$ from the symmetry relation $\operatorname{Re}\lambda_2 = -\operatorname{Re}\lambda_1$. The uncertainty of $\operatorname{Im}\lambda_{1,2}$ does not influence the KS entropy $h = \{\operatorname{Re}\lambda\}_{\max}$.

From the expressions for R and μ, it follows that for

$$T > \tilde{T}_c = T_c + \frac{\pi\chi}{4{\omega_p}^2}\sigma^2,$$



(d) $\mu = 0.9$.

Figure 1. (cont.)

the Kolmogorov-Sinay entropy

$$h = \frac{\sqrt{2}\omega_p}{\sigma}\sqrt{\frac{T-\tilde{T}_c}{T-T_c}}$$

varies quickly from its value at $T < T_c$ which can be taken to be zero, to the finite value of $\sqrt{2}\omega_p/\sigma$. It looks similar to the fast variation of the ordinary thermodynamic entropy at the point of a first-kind phase transition [7].

3. ANALYSIS OF NUMERICAL RESULTS

In this section, the results of numerical analysis of equations (6) are presented for $C_1 = C_2 = 0$ in the (R, μ) parameter plane. We used the standard fourth-order Runge-Kutta algorithm and Hamming's fourth-order modified predictor-corrector method for numeric simulation. Physically meaningful is only the part of the first quadrant ($R > 0$, $\mu > 0$) defined by the condition $\mu > R$.

Figures 1a–1d show the three-dimensional phase portrait, the spectrum of realization, and the projection of Poincare section of the phase manifold with the four-dimensional plane

$$\sum_{i=1}^{4} s_i y_i + d = 0; \qquad y_1 = u, \quad y_2 = u', \quad y_3 = w, \quad y_4 = w',$$

where s_i and d are constants, onto the two-dimensional (y_1, y_2) plane for $R = 0.5$ and various μ values. In the region where $\mu \gg 1.5$, the phase trajectory lies on a manifold which resembles a torus ("quasitorus") (Figures 1a and 1b). Two incommensurable frequencies and their harmonics are present in the realization spectrum in this case. An additional characteristic is the stable presence of the noise component with the intensity increasing while approaching $\mu = 1.5$. If one neglects this noise component, the picture resembles that of torus destruction, as μ approaches the value of 1.5, followed by the transition to dynamical chaos (Figure 1c). The tangling phase trajectory is clearly observed for the three-dimensional phase portrait and Poincare sections at $\mu < 1.5$ (Figure 1d). In this case, the phase trajectories diverge, hindering the retrieval of numerical results. It should be noted that we did not succeed in any computational experiments to observe solutions corresponding to the motion of phase trajectory close to the limit cycle, that agrees with the presence of the noise component. On the other hand, it means that the point of the accumulation of the set of bifurcation points mentioned above corresponds to the arising the quasiperiodic solution of the KdVB equation.

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